Is an arbitrary diffused Borel probability measure in a Polish space without isolated points Haar measure?

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Abstract: It is introduced a certain approach for equipment of an arbitrary set of the cardinality of the continuum by structures of Polish groups and two-sided (left or right) invariant Haar measures. By using this approach we answer positively Maleki's certain question(2012) *what are the real k-dimensional manifolds with at least two different Lie group structures that have the same Haar measure.* It is demonstrated that for each diffused Borel probability measure μ defined in a Polish space $(G, \rho, \mathcal{B}_\rho(G))$ without isolated points there exist a metric ρ_1 and a group operation $\odot$ in G such that $\mathcal{B}_\rho(G) = \mathcal{B}_{\rho_1}(G)$ and $(G, \rho_1, \mathcal{B}_{\rho_1}(G), \odot)$ stands a compact Polish group with a two-sided (left or right) invariant Haar measure μ, where $\mathcal{B}_\rho(G)$ and $\mathcal{B}_{\rho_1}(G)$ denote Borel σ algebras of subsets of G generated by metrics ρ and ρ_1, respectively. Similar result is obtained for construction of locally compact non-compact or non-locally compact Polish groups equipped with two-sided (left or right) invariant quasi-finite Borel measures.

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1. Inroduction

Let $(G, \rho, \odot)$ be a Polish group, by which we mean a group with a complete separable metric ρ for which the transformation (from $G \times G$ onto G) sending (x, y) into $x^{-1} \odot y$ is continuous.

Let $B_\rho(G)$ be the σ-algebra of Borel subsets of G defined by the metric ρ.

The following problem was under intensive consideration by many mathematicians exactly one century ago.

Problem 1.1. Let $(G, \rho, \odot)$ be a locally compact Polish group which is dense-in-itself [1], that is, a space homeomorphic to a separable complete metric space and G has no isolated points. Does there exist a Borel measure μ in $(G, \rho, \odot)$ satisfying the following properties:

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[1] A subset A of a topological space is said to be dense-in-itself if A contains no isolated points.

(i) The measure μ is diffused, that is, μ vanishes on all singletons;

(ii) The measure μ is a two-sided (left or right) invariant, that is, $\mu(g_1 \odot E \odot g_2) = \mu(E)(\mu(g_1 \odot E) = \mu(E)$ or $\mu(E \odot g_2) = \mu(E)$) for every $g_1, g_2 \in G$ and every Borel set $E \in B(G)$;

(iii) The measure μ is outer regular, that is,

$$(\forall E)(E \in B(G) \rightarrow \mu(E) = \inf\{\mu(U) : E \subseteq U \ \& \ U \text{is open}\});$$

(iv) The measure μ is inner regular, that is,

$$(\forall E)(E \in B(G) \rightarrow \mu(E) = \sup\{\mu(F) : F \subseteq E \ \& \ F \text{ is compact}\});$$

(v) The measure μ is finite on every compact set, that is $\mu(K) < \infty$ for all compact K.

The special case of a left (or right) invariant measure for second countable [2] locally compact groups had been shown by Haar in 1933 [8]. Notice that each Polish space is second countable which implies that the answer to Problem 1.1 is yes. The measure μ satisfying conditions (i)-(v) is called a left (right or two-sided) invariant Haar measure in a locally compact Polish group $(G, \rho, \odot)$.

In this note we would like to study the following problems, which can be considered as converse (in some sense) to Problem 1.1.

Problem 1.2. Let (G, ρ) be a Polish metric space which is dense-in-itself. Let μ be a diffused Borel probability measure defined in (G, ρ). Do there exist a metric ρ_1 and a group operation $\odot$ in G such that the following three conditions

(j) The class of Borel measurable subsets of G generated by the metric ρ_1 coincides with the class of Borel measurable subsets of the same space generated by the metric ρ,

(jj) $(G, \rho_1, \odot)$ is a compact Polish group

and

(jjj) μ is a left(right or two-sided) invariant Haar measure in $(G, \rho_1, \odot)$

hold true ?

Problem 1.3. Let (G, ρ) be a Polish metric space which is dense-in-itself. Let μ be a diffused σ-finite non-finite Borel measure defined in (G, ρ). Do there exist a metric ρ_φ, a group operation $\odot_\varphi$ in G and the Borel measure $\mu^\star$ in G such that the following four conditions

(i) The class of Borel measurable subsets of G generated by the metric ρ_φ coincides with the class of Borel measurable subsets of the same space generated by the metric ρ,

(ii) $(G, \rho_\varphi, \odot_\varphi)$ is a non-compact locally compact Polish group,

(iii) The measures $\mu^\star$ and μ are equivalent

and

(iv) $\mu^\star$ is a left (right or two-sided) invariant σ-finite non-finite Haar measure in $(G, \rho_\varphi, \odot_\varphi)$

[2] A topological space T is second countable if there exists some countable collection $\mathcal{U} = \{U_i\}_{i \in N}$ of open subsets of T such that any open subset of T can be written as a union of elements of some subfamily of $\mathcal{U}$

hold true?

Problem 1.4. Let (G, ρ) be a Polish metric space which is dense-in-itself. Let μ be a diffused non-σ-finite quasi-finite Borel measure defined in (G, ρ). Do there exist a metric ρ_1 and a group operation $\odot$ in G such that the following three conditions

(j) The class of Borel measurable subsets of G generated by the metric ρ_1 coincides with the class of Borel measurable subsets of the same space generated by the metric ρ,

(jj) $(G, \rho_1, \odot)$ is a non-locally compact Polish group

and

(jjj) μ is a left(right or two-sided) invariant quasi-finite Borel measure in $(G, \rho_1, \odot)$

hold true ?

In [4], the author uses methods of the theory ultrafilters to present a modified proof that a locally compact group with a countable basis has a left invariant and right invariant Haar measure. The author first shows that the topological space $(\beta_1 X; \tau_1)$ consisting of all ultrafilters on a non-empty set X is homeomorphic to the topological space $(\beta_2 X; \tau_2)$ of all nonzero multiplicative functions in the first dual space $\ell^*_\infty(X)$ (Theorem 3.8). By using this result the author proves the existence of the infinitely additive left invariant measure λ on compact sets of the locally compact Hausdorff topological group G (Theorem 7.1). Starting from this point, the author introduces the notion of ν-measurable subsets in G where ν is an outer measure in G induced by the λ and open sets in G, and proves the existence of a left invariant Haar measure by the scheme presented in [7]. Notice that his proof essentially uses the axiom of choice. Several examples of the Haar measure are presented. It is underlined by Example 9.7 that $G = R^k$ with $k = \frac{n^2-n}{2}$ has two Lie group structures but the Lebesgue measure on R^k is the Haar measure on both Lie groups. In this context the following question was stated in this paper.

Problem 1.5([4],Question 9.8) What are the real k-dimensional manifolds with at least two different Lie group structures that have the same Haar measure?

The rest of the paper is the following.

In Section 2 we introduce a certain approach for equipment of an arbitrary set of the cardinality of the continuum by structures of various(compact, locally compact or non-locally compact) Polish groups with two-sided(left or right) invariant Borel measures and study Problem 1.5.

In Section 3 we study general question whether an arbitrary diffused Borel probability measure in a Polish space without isolated points is Haar measure and give its affirmative resolution. Moreover, we study Problems 1.2, 1.3 and answer to them positively.

2. Equipment of an arbitrary set of the cardinality of the continuum by structures of Polish groups

Theorem 2.1. *Let X be a set of the cardinality of the continuum and $(G, \odot, \rho)$ a Polish group. Further, let $f : G \to X$ be a one-to-one mapping. We set*

$$x \odot_f y = f(f^{-1}(x) \odot f^{-1}(y))$$

and

$$\rho_f(x, y) = \rho(f^{-1}(x), f^{-1}(y))$$

for $x, y \in X$. Then the following conditions hold true:

(i) $(G_f, \odot_f, \rho_f)$ is a Polish group which is Borel isomorphic to the Polish group $(G, \odot, \rho)$;

(ii) If $(G, \odot, \rho)$ is an abelian Polish group then so is $(G_f, \odot_f, \rho_f)$;

(iii) If ρ is two-sided invariant metric in $(G, \odot)$ so is ρ_f in $(G_f, \odot_f)$;

(iv) If $(G, \odot, \rho)$ is dense-in-itself so is $(X, \odot_f, \rho_f)$;

(v) If $(G, \odot, \rho)$ is a compact Polish group then so is $(X, \odot_f, \rho_f)$;

(vi) If $(G, \odot, \rho)$ is a locally compact Polish group then so is $(X, \odot_f, \rho_f)$;

(vii) If $(G, \odot, \rho)$ a non-locally compact Polish group then so is $(X, \odot_f, \rho_f)$;

(viii) If $(G, \odot, \rho)$ is a locally compact or compact Polish group and λ is a left(or right or two-sided) invariant Haar measure in $(G, \odot, \rho)$, then λ_f also is a left(or right or two-sided) invariant Haar measure in $(G_f, \odot_f, \rho_f)$, where $G_f = X$, $\mathcal{B}_{\rho_f}(G_f)$ is Borel σ-algebra of G_f generated by the metric ρ_f and λ_f is a Borel measure in G_f defined by

$$(\forall Y)(Y \in \mathcal{B}_f(G_f) \to \lambda_f(Y) = \lambda(f^{-1}(Y))).$$

(ix) If $(G, \odot, \rho)$ is a non-locally compact Polish group and λ is a left(or right or two-sided) invariant quasi-finite [3] Borel measure in $(G, \odot, \rho)$, then λ_f also is a left(or right or two-sided) invariant quasi-finite Borel measure in $(G_f, \odot_f, \rho_f)$, where $G_f = X$, $\mathcal{B}_{\rho_f}(G_f)$ is Borel σ-algebra of G_f generated by the metric ρ_f and λ_f is a Borel measure in G_f defined by

$$(\forall Y)(Y \in \mathcal{B}_f(G_f) \to \lambda_f(Y) = \lambda(f^{-1}(Y))).$$

Proof. **Proof of the item (i).**

Closure . If $x, y \in X$ then $x \odot_f y = f(f^{-1}(x) \odot f^{-1}(y)) \in X$.

Associativity . For all x, y and z in X, we have

$$(x \odot_f y) \odot_f z = f[f^{-1}(x \odot_f y) \odot f^{-1}(z)] = f[f^{-1}(f(f^{-1}(x) \odot f^{-1}(y))) \odot f^{-1}(z)] =$$

$$f[(f^{-1}(x) \odot f^{-1}(y)) \odot f^{-1}(z)] = f[f^{-1}(x) \odot (f^{-1}(y) \odot f^{-1}(z))] =$$

$$f[f^{-1}(x) \odot f^{-1}(y \odot_f z)] = x \odot_f (y \odot_f z).$$

[3] A measure μ is called quasi-finite if there is a μ-measurable set X with $0 < \mu(X) < +\infty$.

Identity element. Let e be an identity element of G. Setting $e_f := f(e) \in X$, for $x \in X$ we have

$$x \odot_f e_f = x \odot_f f(e) = f(f^{-1}(x) \odot f^{-1}(f(e))) = f(f^{-1}(x) \odot e) = f(f^{-1}(x)) = x$$

and

$$e_f \odot_f x = f(e) \odot_f x = f(f^{-1}(f(e)) \odot f^{-1}(x)) = f(e \odot f^{-1}(x)) = f(f^{-1}(x)) = x.$$

The latter relations means that e_f is the identity element of X.

Inverse element. If $a \in G$ then we denote its inverse element by a_G^{-1}. For $x \in X$ setting $x_X^{-1} = f((f^{-1}(x))_G^{-1})$, we have

$$x \odot_f x_X^{-1} = f(f^{-1}(x) \odot f^{-1}(x_X^{-1})) = f(f^{-1}(x) \odot f^{-1}(f((f^{-1}(x))_G^{-1}))) =$$
$$f(f^{-1}(x) \odot (f^{-1}(x))_G^{-1}) = f(e) = e_f$$

and

$$x_X^{-1} \odot_f x = f(f^{-1}(x_X^{-1}) \odot f^{-1}(x)) = f(f^{-1}(f((f^{-1}(x))_G^{-1})) \odot f^{-1}(x)) =$$
$$f((f^{-1}(x))_G^{-1} \odot f^{-1}(x)) = f(e) = e_f.$$

The latter relations means that x_X^{-1} is an inverse element of x.

Continuity of the operation $(x, y) \to x \odot_f y_X^{-1}$ when $(a, b) \to a \odot b_G^{-1}$ is continuous.

For all neighbourhood $U_X(x \odot_f y_X^{-1}, r)$ we have to choose such neighbourhoods $U_X(x, r_1)$ and $U_X(y, r_2)$ of elements x and y respectively that $(w_1 \odot_f (w_2)_X^{-1}) \in U_X(x \odot_f y_X^{-1}, r)$ for $w_1 \in U_X(x, r_1)$ and $w_2 \in U_X(y, r_2)$.

We have

$$U_X(x \odot_f y_X^{-1}, r) = \{z : \rho_f(z, x \odot_f y_X^{-1}) < r\} = \{z : \rho(f^{-1}(z), f^{-1}(x \odot_f y_X^{-1})) < r\} =$$
$$\{z : \rho(f^{-1}(z), f^{-1}(f(f^{-1}(x) \odot f^{-1}(y_X^{-1})))) < r\} =$$
$$\{z : \rho(f^{-1}(z), f^{-1}(f(f^{-1}(x) \odot f^{-1}(f((f^{-1}(y))_G^{-1}))))) < r\} =$$
$$\{z : \rho(f^{-1}(z), (f^{-1}(x) \odot (f^{-1}(y))_G^{-1})))) < r\}.$$

Since $(a, b) \to a \odot b_G^{-1}$ is continuous, for $a = f^{-1}(x)$, $b = f^{-1}(y)$ and $r > 0$ we can choose such neighbourhoods $U_G(f^{-1}(x), r_1)$ and $U_G(f^{-1}(y), r_2)$ of elements $f^{-1}(x)$ and $f^{-1}(y)$ respectively that then $(a_1 \odot (a_2)_G^{-1}) \in U_G(f^{-1}(x) \odot (f^{-1}(y))_G^{-1}, r)$ for $a_1 \in U_G(f^{-1}(x), r_1)$ and $a_2 \in U_G(f^{-1}(y), r_2)$.

It is obvious to check the validity of the following equalities

$$U_X(x, r_1) = f(U_G(f^{-1}(x), r_1)),$$

$$U_X(y, r_2) = f(U_G(f^{-1}(y), r_2)),$$

$$U_X(x \odot_f y_X^{-1}, r) = f(U_G(f^{-1}(x) \odot (f^{-1}(y))_G^{-1}, r).$$

Notice that if $w_1 \in U_X(x, r_1)$ and $w_2 \in U_X(y, r_2)$ then $(w_1 \odot_f (w_2)_X^{-1}) \in U_X(x \odot_f y_X^{-1}, r)$. Indeed,$w_1 \in U_X(x, r_1)$ and $w_2 \in U_X(y, r_2)$ imply that $f^{-1}(w_1) \in$

$U_G(f^{-1}(x),r_1)$ and $f^{-1}(w_2)\in U_G(f^{-1}(y),r_2)$ from which we deduce that $(w_1\odot (w_2)_G^{-1})\in U_G(f^{-1}(x)\odot(f^{-1}(y))_G^{-1},r)$.

Borel isomorphism of $(G,\odot,\rho)$ and $(G_f,\odot_f,\rho_f)$. Notice that this isomorphism is realized by the mapping $f:G\rightarrow G_f$.

Proof of the item (ii).

Since $(G,\odot)$ is an abelian Polish group, for $x,y\in G_f$ we have

$$x\odot_f y=f(f^{-1}(x)\odot f^{-1}(y))=f(f^{-1}(y)\odot f^{-1}(x))=y\odot_f x.$$

Proof of the item (iii).

Since ρ is a two-sided invariant metric in $(G,\odot)$ we have $\rho(h_1\odot x\odot h_2,h_1\odot y\odot h_2)=\rho(x,y)$ for each $x,y,h_1,h_2\in G$. Take into account this fact and the associativity property of the group operation $\odot_f$, we get that the condition

$$\rho_f(h_1^*\odot_f x^*\odot_f h_2^*,h_1^*\odot_f y^*\odot_f h_2^*)=$$
$$\rho_f(f(f^{-1}(h_1^*)\odot f^{-1}(x^*)\odot f^{-1}(h_2^*)),f(f^{-1}(h_1^*)\odot f^{-1}(y^*)\odot f^{-1}(h_2^*)))=$$
$$\rho(f^{-1}(h_1^*)\odot f^{-1}(x^*)\odot f^{-1}(h_2^*),f^{-1}(h_1^*)\odot f^{-1}(y^*)\odot f^{-1}(h_2^*))=$$
$$\rho(f^{-1}(x^*),f^{-1}(y^*))=\rho_f(f(f^{-1}(x^*)),f(f^{-1}(y^*)))=\rho_f(x^*,y^*)$$

holds true for each $x^*,y^*,h_1^*,h_2^*\in G_f$.

Proof of the item (iv). We have to show that if $(G,\odot,\rho)$ is dense-in-itself then so is $(G_f,\odot_f,\rho_f)$. Indeed assume the contrary and let x^* be an isolated point of G_f. The latter relation means that for some $\epsilon>0$ we have $\rho_f(y^*,x^*)\geq\epsilon$ for each $y^*\in G_f\setminus\{x^*\}$ which implies that $\rho(y,x)\geq\epsilon$ for each $y\in G\setminus\{x\}$ where $x=f^{-1}(x^*)$. We get the contradiction and the validity of the item (iv) is proved.

Proof of the item (v). We have to prove that if a family of open sets $(U_i^*)_{i\in I}$ whose union covers the space G_f then there is its subfamily whose union also covers the same space. Let consider a family of sets $(f^{-1}(U_i^*))_{i\in I}$. Since it is the family of open sets whose union covers the space G and G is a compact space, we claim that there is a finite subfamily $(f^{-1}(U_{i_k}^*))_{1\leq k\leq n}(i_k\in I$ for $k=1,\cdots,n))$ whose union $\cup_{k=1}^n f^{-1}(U_{i_k}^*)$ covers G. Now it is obvious that the family $(U_{i_k}^*)_{1\leq k\leq n}$ is the family of open sets (in G_f)) whose union also covers G_f.

Proof of the item (vi). Let $x^*\in G_f$. Since $(G,\odot,\rho)$ is locally compact the point $f^{-1}(x^*)$ has a compact neighbourhood U. Now it is obvious that the set $f(U)$ will be a compact neighbourhood of the point x^*. Since $x^*\in G_f$ was taken arbitrary the validity of the item (vi) is proved.

Proof of the item (vii). Since $(G,\odot,\rho)$ is no locally compact there is a point x_0 which has no a compact neighbourhood. Now if we consider a point $f(x_0)$, we observe that it has no a compact neighbourhood. Indeed, if assume the contrary and U is a compact neighbourhood of the point $f(x_0)$ then $f^{-1}(U)$ also will be a compact neighbourhood of the point x_0 and we get the contradiction. This ends the proof of the item (vii).

Proof of the item (viii).

Proof of the diffusivity of the measure λ_f. Since λ vanishes on all singletons, we have

$$\lambda_f(x) = \lambda(f^{-1}(x))) = 0$$

for each $x \in G_f$;

Proof of the left(or right or two-sided) invariance of the measure λ_f. If $(G, \odot, \rho)$ is a locally compact or compact Polish group and λ is a left(or right or two-sided) invariant Haar measure in $(G, \odot, \rho)$, then λ_f also will be a left(or right or two-sided) invariant Haar measure in $(G_f, \odot_f, \rho_f, \mathcal{B}_{\rho_f}(G_f))$, where $G_f = X$, $\mathcal{B}_{\rho_f}(G_f)$ is Borel σ-algebra of G_f generated by the metric ρ_f and λ_f is defined by

$$(\forall Y)(Y \in \mathcal{B}_{\rho_f}(G_f) \rightarrow \lambda_f(Y) = \lambda(f^{-1}(Y))).$$

Case 1. λ is a left invariant Haar measure in $(G, \odot, \rho)$.

$$(\forall Y)(\forall h)((Y \in \mathcal{B}_{\rho_f}(G_f) \ \&\ h \in G_f) \rightarrow \lambda_f(h \odot_f Y) =$$
$$\lambda(f^{-1}(h \odot_f Y)) = \lambda(f^{-1}(h) \odot f^{-1}(Y)) = \lambda(f^{-1}(Y)) = \lambda_f(Y))).$$

Case 2. λ is a right invariant Haar measure in $(G, \odot, \rho)$.

$$(\forall Y)(\forall h)((Y \in \mathcal{B}_{\rho_f}(G_f) \ \&\ h \in G_f) \rightarrow \lambda_f(Y \odot_f h) =$$
$$\lambda(f^{-1}(Y \odot_f h)) = \lambda(f^{-1}(Y) \odot f^{-1}(h)) = \lambda(f^{-1}(Y)) = \lambda_f(Y))).$$

Case 3. λ is a two-sided invariant Haar measure in $(G, \odot, \rho)$.

$$(\forall Y)(\forall h_1)(\forall h_2)((Y \in \mathcal{B}_{\rho_f}(G_f) \ \&\ h_1 \in G_f) \ \&\ h_2 \in G_f) \rightarrow \lambda_f(h_1 \odot_f Y \odot_f h_2) =$$
$$\lambda(f^{-1}(h_1 \odot_f Y \odot_f h_2)) = \lambda(f^{-1}(h_1) \odot f^{-1}(Y) \odot f^{-1}(h_2)) = \lambda(f^{-1}(Y)) = \lambda_f(Y))).$$

Proof of the outer regularity of the measure λ_f. Let take any set $E_f \in B_{\rho_f}(G_f)$ and any $\epsilon > 0$. Let consider a set $f^{-1}(E_f) \in B(G)$. Since λ is outer regular there is an open subset U of G such that $f^{-1}(E_f) \subseteq U$ and $\lambda(U \setminus f^{-1}(E_f)) < \epsilon$. Then we get

$$\lambda_f(f(U) \setminus E_f) = \lambda(f^{-1}(f(U) \setminus E_f)) = \lambda(U \setminus f^{-1}(E_f)) < \epsilon.$$

Proof of the inner regularity of the measure λ_f. Let take any set $E_f \in B_{\rho_f}(G_f)$ and any $\epsilon > 0$. Let consider a set $f^{-1}(E_f) \in B(G)$. Since λ is inner regular there is a compact subset F of G such that $F \subseteq f^{-1}(E_f)$ and $\lambda(f^{-1}(E_f) \setminus F) < \epsilon$. Then we get

$$\lambda_f(E_f \setminus f(F)) = \lambda(f^{-1}(E_f \setminus f(F)) = \lambda(f^{-1}(E_f) \setminus F) < \epsilon.$$

Proof of the finiteness of the measure λ_f on all compact subsets. Let take any compact set $F \subseteq G_f$. Since $f^{-1}(F)$ is compact in G and the measure λ is finite on every compact set we get $\lambda_f(F) = \lambda(f^{-1}(F)) < \infty$.

Proof of the item (ix). The proof of this item can be obtained by the scheme used in the proof of the item (viii).

□

Below we consider some examples which employ the constructions described by Theorem 2.1.

Example 2.1. *Let $f : R \to (-c,c)$ be defined by $f(y) = \frac{c(e^y-1)}{1+e^y}$ for $y \in R$, where $c > 0$. Then $f^{-1} : (-c,c) \to R$ is defined by $f^{-1}(x) = \ln(\frac{c+x}{c-x})$ for $x \in (-c,c)$. For $x, y \in (-c,c)$ we put*

$$x +_f y = f(f^{-1}(x)+f^{-1}(y)) = f(\ln(\frac{c+x}{c-x})+\ln(\frac{c+y}{c-y})) = f(\ln(\frac{(c+x)(c+y)}{(c-x)(c-y)})) =$$

$$\frac{c(e^{\ln(\frac{(c+x)(c+y)}{(c-x)(c-y)})}-1)}{1+e^{\ln(\frac{(c+x)(c+y)}{(c-x)(c-y)})}} = \frac{c(\frac{(c+x)(c+y)}{(c-x)(c-y)}-1)}{1+\frac{(c+x)(c+y)}{(c-x)(c-y)}} =$$

$$c\frac{(c+x)(c+y)-(c-x)(c-y)}{(c-x)(c-y)+(c+x)(c+y)} = c\frac{2cx+2cy}{2c^2+2xy} = \frac{x+y}{1+\frac{xy}{c^2}}.$$

Note that λ_f defined by

$$(\forall Y)(Y \in \mathcal{B}_{\rho_f}((-c,c)) \to \lambda_f(Y) = \lambda(\{\ln(\frac{c+y}{c-y}) : y \in Y\}) = \int_Y \frac{c^2}{c^2-t^2}dt)$$

will be Haar measure in $(-c,c)$, where λ denotes a linear Lebesgue measure in R.

Remark 2.1. *Example 2.2 demonstrates that the Haar measure space $(G_c, \star, \rho_{G_c}, \nu)$ which comes from [4](cf. Example 9.1, p.61) exactly coincides with a Polish group $(R_f, +_f, \rho_f, \lambda_f)$ where ρ is a usual metric in R, λ is a linear Lebesgue measure in R and $f : R \to (-c,c)$ is a mapping defined by $f(y) = \frac{c(e^y-1)}{1+e^y}$ for $y \in R$.*

It is well known(see, [5], Eq. 35, p. 5) that the relativistic law of adding velocities has the following form

$$v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$$

for $v_1, v_2 \in (-c,c)$, where c denotes the speed of light. This operation of adding exactly coincides with the operation $+_f$ under which $(-c,c)$ stands a locally compact non-compact Polish group. Hence the Haar measure λ_f can be used in studding properties of the inertial reference frame O_0 which moves relative to O with velocity v in along the x axis (see, [5], p. 1).

Example 2.2. *Let $(R, +, \rho)$ be a one-dimensional Euclidian vector space and λ a linear Lebesgue measure in R. Let $f : R \to (0,+\infty)$ be defined by $f(x) = e^x$. We put*

$$x +_f y = exp\{\ln(x)+\ln(y)\} = exp\{\ln(xy)\} = xy$$

and

$$\rho_f(x,y) = |\ln(x) - \ln(y)|$$

for $x, y \in (0, +\infty)$. *We define* λ_f *by*

$$(\forall Y)(Y \in \mathcal{B}((0,+\infty)) \rightarrow \lambda_f(Y) = \lambda(\{\ln(y) : y \in Y\})).$$

By Theorem 2.1 we know that λ_f *is Haar measure in* $(0, +\infty)$. *Since*

$$(\forall Z)(Z \in \mathcal{B}(\ R) \rightarrow \lambda(Z) = \int_Z dx)$$

we deduce that

$$(\forall Y)(Y \in \mathcal{B}((0,+\infty)) \rightarrow \lambda_f(Y) = \lambda(\ln(Y)) = \int_{\ln(Y)} dx = \int_Y \frac{dx}{x}).$$

Note that Haar measure space $(G, \cdot, \rho_G, \nu)$ *constructed in [4](see p.54) coincides with Haar measure space* $(\ R_f, +_f, \rho_f, \lambda_f)$.

Example 2.3. *Let* $X = (-c, c)$ *where* $c > 0$. *We define* $f :\ R \rightarrow (-c, c)$ *by* $f(x) = \frac{2carctg(x)}{\pi}$ *for* $x \in\ R$. *Then* $f^{-1}(w) = tg(\frac{\pi w}{2c})$ *for* $w \in (-c, c)$. *We have*

$$x \odot_f y = f(f^{-1}(x) \odot f^{-1}(y)) = f(tg(\frac{\pi x}{2c}) + tg(\frac{\pi y}{2c})) = \frac{2carctg(tg(\frac{\pi x}{2c}) + tg(\frac{\pi y}{2c}))}{\pi} =$$

and

$$\rho_f(x, y) = \rho(f^{-1}(x), f^{-1}(y)) = |tg(\frac{\pi x}{2c}) - tg(\frac{\pi y}{2c})|.$$

for $x, y \in (-c, c)$.

Then we get a new example of Haar measure space $(\ R_f, +_f, \rho_f, \lambda_f)$. *Note that the Haar measure* λ_f *in* $(-c, c)$ *is defined by*

$$(\forall Y)(Y \in \mathcal{B}_f((-c,c)) \rightarrow \lambda_f(Y) = \lambda(f^{-1}(Y))) = \lambda(\{tg(\frac{\pi w}{2c}) : w \in Y\})).$$

Example 2.4. *Let* $f :\ R \rightarrow\ Z \times \{0, 1, \cdots, 9\}^{\ N}$ *be defined by* $f(a_0 + 0, a_1 a_2 \cdots) = (a_0, a_1, a_2, \cdots)$ *for* $a_0 \in\ Z$ *and* $(a_0, a_1, a_2, \cdots) \in \{0, 1, \cdots, 9\}^{\ N}$.

Then $f^{-1} :\ Z \times \{0, 1, \cdots, 9\}^{\ N} \rightarrow\ R$ *is defined by* $f^{-1}((a_0, a_1, a_2, \cdots)) = a_0 + 0, a_1 a_2 \cdots$. *We put*

$$(a_0, a_1, a_2, \cdots) +_f (b_0, b_1, b_2, \cdots) = f(f^{-1}((a_0, a_1, a_2, \cdots)) + f^{-1}((b_0, b_1, b_2, \cdots))) =$$

$$f(a_0, a_1 a_2 \cdots + b_0, b_1 b_2 \cdots) = f(c_0, c_1 c_2 \cdots) = (c_0, c_1, c_2 \cdots),$$

where $c_0, c_1 c_2 \cdots = a_0, a_1 a_2 \cdots + b_0, b_1 b_2 \cdots$. *The metric* ρ_f *in* $\ Z \times \{0, 1, \cdots, 9\}^{\ N}$ *is defined by*

$$\rho_f((a_0, a_1, a_2, \cdots), (b_0, b_1, b_2, \cdots)) = \rho(f^{-1}((a_0, a_1, a_2, \cdots)), f^{-1}((b_0, b_1, b_2, \cdots))) =$$

$$\rho(a_0 + 0, a_1 a_2 \cdots, b_0 + 0, b_1 b_2 \cdots) = |(a_0 + 0, a_1 a_2 \cdots) - (b_0 + 0, b_1 b_2 \cdots)|.$$

By Theorem 2.1 we know that λ_f *defined by*

$$(\forall Y)(Y \in \mathcal{B}_{\rho_f}(Z \times \{0,1,\cdots,9\}^{N}) \rightarrow \lambda_f(Y) = \lambda(f^{-1}Y) =$$

$$\lambda(\{a_0, a_1a_2 \cdots : (a_0, a_1, a_2, \cdots) \in Y)$$

is Haar measure in $Z \times \{0,1,\cdots,9\}^{N}$*, where* λ *denotes a linear Lebesgue measure in* R*.*

Remark 2.2. *Let* M *be a topological space. A homeomorphism* $\phi : U \rightarrow V$ *of an open set* $U \subseteq M$ *onto an open set* $V \subseteq R^d$ *will be called a local coordinate chart (or just 'a chart') and* U *is then a coordinate neighbourhood (or 'a coordinate patch') in* M*.*

A C^∞ *differentiable structure, or smooth structure, on* M *is a collection of coordinate charts* $\phi_\alpha : U_\alpha \rightarrow V_\alpha \subseteq R^d$ *(same d for all* α*'s) such that*

(i) $M = \cup_{\alpha \in A} U_\alpha$;

(ii) any two charts are 'compatible': for every α, β *the change of local coordinates* $\phi_\beta \circ \phi_\alpha^{-1}$ *is a smooth* C^∞ *map on its domain of definition, i.e. on* $\phi_\alpha(U_\beta \cap U_\alpha) \subseteq R^d$*;*

(iii) the collection of charts ϕ_α *is maximal with respect to the property (ii): if a chart* ϕ *of* M *is compatible with all* ϕ_α *then* ϕ *is included in the collection.*

A topological space equipped with a C^∞ *differential structure is called a real smooth manifold. Then d is called the dimension of* M*,* $d = dimM$*.*

Recall, that a Lie group is a set G *with two structures:* G *is a group and* G *is a real smooth manifold. These structures agree in the following sense: multiplication and inversion are smooth maps.*

In [4](see, Example 9.7, p. 64), it is shown that $G = R^k$ *with* $k = \frac{n^2-n}{2}$ *has two different Lie group structure and the Lebesgue measure in* R^k *is Haar measure on both Lie groups. Further the author asks(see ,[4], Question 9.8) what are real k dimensional manifolds with at least two different Lie group structures that have the same Haar measure.*

The next example answers positively to Maleki's question described in Remark 2.2.

Example 2.5. *For* $n > 2$*, let* $(R^n, \rho_n, +, \lambda_n)$ *be an n-dimensional Euclidean vector space equipped with standard metric* ρ_n *and n-dimensional Lebesgue measure* λ_n*. Let* $f : R^n \rightarrow R^n$ *be defined by* $f(x_1, x_2, x_3, \cdots, x_n) = (x_1, x_1^2 + x_2, x_3, \cdots, x_n)$ *for* $(x_1, x_2, x_3, \cdots, x_n) \in R^n$*.*

It is obvious that

1) f *is bijection of* R^n *and* $f^{-1}((x_1, x_2, x_3, \cdots, x_n)) = (x_1, x_2 - x_1^2, x_3, \cdots, x_n)$ *for* $(x_1, x_2, x_3, \cdots, x_n) \in R^n$*;*

2) f *as well* f^{-1} *is infinitely many times continuously differentiable;*

3) f *is not linear;*

4) f *as well* f^{-1} *preserves Lebesgue measure* λ_n*.*

Let consider $((R^n)_f, (\rho_n)_f, +_f, (\lambda_n)_f)$*. By virtue of Theorem 2.1 we deduce that* $((R^n)_f, (\rho_n)_f, +_f, (\lambda_n)_f)$ *is a locally compact non-compact Polish group with two-sided invariant Haar measure* $(\lambda_n)_f$*.*

Note that $(R^n)_f = R^n$*;*

b) $(\rho_n)_f(x, y) = \rho_n(f^{-1}(x), f^{-1}(y))$*;*

c) $x +_f y = f(f^{-1}(x) + f^{-1}(y))$*;*

Note that the operation $''+_f''$ *is commutative but it differs from the usual addition operation* $''+''$*. Indeed, we have*

$$(1, 1, \cdots, 1) +_f (2, 2, \cdots, 2) = f(f^{-1}(1, 1, \cdots, 1) + f^{-1}(2, 2, \cdots, 2)) =$$

$$f((1, 0, 1, \cdots, 1) + (2, -2, 2, \cdots, 2)) = f(3, -2, 3, \cdots, 3) = (3, 7, 3, \cdots, 3)$$

and

$$(1, \cdots, 1) + (2, \cdots, 2) = (3, \cdots, 3).$$

Since f *is Borel measurable, by using Theorem 2.1 we deduce that* $\mathcal{B}_{\rho_f}(R^n) = \mathcal{B}(R^n)$*.*

Note also that $(\lambda_n)_f = \lambda_n$*. Indeed, by Theorem 2.1 we have that*

$$(\forall Y)(Y \in \mathcal{B}(R^n) \rightarrow \lambda_f(Y) = \lambda(f^{-1}(Y)) = \lambda(Y)).$$

Remark 2.3. *Notice that Example 2.5 extends the result of Example 9.7 [4]. Indeed, it is obvious that for* $n > 2$*, measure space* $((R^n)_f, (\rho_n)_f, +_f, (\lambda_n)_f) = (R^n, (\rho_n)_f, +_f, \lambda_n)$ *has Lie group structure which differs from standard Lie group structure of* R^n *because group operations* $''+''$ *and* $''+_f''$*, as were showed in Example 2.8, are different. Furthermore the Lebesgue measure* λ_n *(in* R^n*) is Haar measure on both Lie groups.*

Now let consider $\ell_2 = \{(x_k)_{k \in N} : x_k \in R \;\&\; k \in N \;\&\; \sum_{k \in N} x_k^2 < \infty\}$ as a vector space with usual addition operation "$+$". If we equip ℓ_2 with standard metric ρ_{ℓ_2} defined by

$$\rho_{\ell_2}((x_k)_{k \in N}, (y_k)_{k \in N}) = \sqrt{\sum_{k \in N} (x_k - y_k)^2}$$

for $(x_k)_{k \in N}, (y_k)_{k \in N} \in \ell_2$, then $(\ell_2, ''+'', \rho_{\ell_2})$ stands an example of a non-locally compact Polish group. Here naturally arise a question asking whether there exists a metric ρ in ℓ_2 such that $\big(\ell_2, "+", \rho\big)$ stands an example of a locally compact σ-compact Polish group. An affirmative answer to this question is containing in the following example.

Example 2.6. *Let consider* R *and* ℓ_2 *as vector spaces over the group of all rational numbers* Q*. Let* $(a_i)_{i \in I}$ *and* $(b_i)_{i \in I}$ *be Hamel bases in* R *and* ℓ_2*, respectively. For* $x \in R \setminus \{0\}$*, there exists a unique sequence of non-zero rational numbers* $(q_{i_k}^{(x)})_{1 \le k \le n_x}$ *such that* $x = \sum_{k=1}^{n_x} q_{i_k}^{(x)} a_{i_k}$*. We set* $f(x) = \sum_{k=1}^{n_x} q_{i_k}^{(x)} b_{i_k}$ *for* $x \in R \setminus \{0\}$ *and* $f(0) = (0, 0, \cdots)$*. Notice that* $f : R \rightarrow \ell_2$ *is one-to-one linear transformation.*

Let $x=\sum_{k=1}^{n_x} q_{i_k}^{(x)} b_{i_k}$ *and* $y=\sum_{k=1}^{n_y} q_{i_k}^{(y)} b_{i_k}$. *Now if we set*

$$x +_f y = f(f^{-1}(x) \odot f^{-1}(y)),$$

then we will obtain

$$x +_f y = f(f^{-1}(x) + f^{-1}(y)) = f(\sum_{k=1}^{n_x} q_{i_k}^{(x)} a_{i_k} + \sum_{k=1}^{n_y} q_{i_k}^{(y)} a_{i_k}) =$$

$$\sum_{k=1}^{n_x} q_{i_k}^{(x)} b_{i_k} + \sum_{k=1}^{n_y} q_{i_k}^{(y)} b_{i_k} = x+y,$$

which means that a group operation $+_f$ *coincides with usual addition operation* "$+$".

Let define ρ *by*

$$\rho(x,y) = |f^{-1}(x) - f^{-1}(y)| = |\sum_{k=1} n_x q_{i_k}^{(x)} a_{i_k} - \sum_{k=1} n_y q_{i_k}^{(y)} a_{i_k}|.$$

By Theorem 2.1 we know that $(R_f, +_f, \rho_f)$, *equivalently* $(\ell_2, +, \rho_f)$ *is a locally compact non-compact Polish group which is isomorphic to the Polish group* $(R, +, |\cdot|)$.

Moreover, if $(R, +, |\cdot|, \lambda)$ *is Haar measure space, then* $(\ell_2, +, \rho_f, \lambda_f)$ *also is Haar measure space. Denoting by* $\mathcal{B}_{\rho_f}(\ell_2)$ *a Borel* σ*-algebra of subsets of* ℓ_2 *generated by the metric* ρ_f, *we define Haar measure* λ_f *in* ℓ_2 *by*

$$(\forall Y)(Y \in \mathcal{B}_{\rho_f}(\ell_2) \to \lambda_f(Y) = \lambda(f^{-1}(Y))).$$

Remark 2.4. *Let* $(G, \rho, +)$ *be an abelian Polish group. We say that* G *is one-dimensional group w.r.t. metric* ρ *if for each* $n \in N$ *and for each family of different elements* $(a_k)_{1 \le k \le n}$ *there is permutation* h *of* $\{1, 2, \cdots, n\}$ *such that*

$$\rho(a_{h(1)}, a_{h(n)}) = \sum_{k=1}^{n-1} \rho(a_{h(k)}, a_{h(k+1)}).$$

Then it is obvious to show that $(\ell_2, +, \rho_f, \lambda_f)$ *is one-dimensional group w.r.t. metric* ρ_f.

Example 2.7. *Let consider* R^∞ *and* R *as vector spaces over the group of all rational numbers* Q. *Let* $(a_i)_{i \in I}$ *and* $(b_i)_{i \in I}$ *be Hamel bases in* R^∞ *and* R, *respectively. For* $x \in R^\infty \setminus \{(0,0,\cdots)\}$, *there exists a unique sequence of non-zero rational numbers* $(q_{i_k}^{(x)})_{1 \le k \le n_x}$ *such that* $x = \sum_{k=1}^{n_x} q_{i_k}^{(x)} a_{i_k}$. *We set* $f(x) = \sum_{k=1}^{n_x} q_{i_k}^{(x)} b_{i_k}$ *for* $x \in R^\infty \setminus \{(0,0,\cdots)\}$ *and* $f(0,0,\cdots) = 0$. *Notice that* $f: R^\infty \to R$ *is one-to-one linear transformation.*

For $w, z \in R$, *setting*

$$w +_f z = f(f^{-1}(w) + f^{-1}(z)),$$

we get

$$w +_f z = f(f^{-1}(w) + f^{-1}(z)) = f(\sum_{k=1}^{n_w} q_{i_k}^{(w)} a_{i_k} + \sum_{k=1}^{n_z} q_{i_k}^{(z)} a_{i_k}) =$$

$$\sum_{k=1}^{n_w} q_{i_k}^{(w)} b_{i_k} + \sum_{k=1}^{n_z} q_{i_k}^{(z)} b_{i_k} = w + z,$$

which means that a group operation $+_f$ coincides with usual addition operation $''+''$ in R.

Let define ρ by

$$\rho(w, z) = \rho_T(f^{-1}(w), f^{-1}(z)),$$

where ρ_T is Tychonov metric in R^∞ defined by

$$\rho_T((x_k)_{k\in N}, (y_k)_{k\in N}) = \sum_{k=1}^{\infty} \frac{|x_k - y_k|}{2^k(1 + |x_k - y_k|)}$$

for $(x_k)_{k\in N}, (y_k)_{k\in N} \in R^\infty$.

By Theorem 2.1 we know that $(R_f^\infty, +_f, \rho_f)$, equivalently, $(R, +, \rho_f)$ is an abelian non-locally compact Polish group which is isomorphic to the abelian non-locally compact Polish group $(R^\infty, +, \rho_T)$.

Let λ be a translation invariant quasifinite borel measure in R^∞(see, for example, [1], [2]).

We put

$$(\forall Y)(Y \in \mathcal{B}_{\rho_f}(R) \to \lambda_f(Y) = \lambda(f^{-1}(Y))).$$

Since λ is translation invariant quasifinite borel measure in R^∞, by virtue of Theorem 2.1 we deduce that so is the measure λ_f in $(R, +, \rho_f)$.

3. Is an arbitrary diffused Borel probability measure in a Polish space Haar measure?

The following lemma is a useful ingredient for our further investigations.

Lemma 3.1. *Let E_1 and E_2 be any two Polish topological spaces without isolated points. Let μ_1 be a probability diffused Borel measure on E_1 and let μ_2 be a probability diffused Borel measure on E_2. Then there exists a Borel isomorphism $\varphi : (E_1, B(E_1)) \to (E_2, B(E_2))$ such that*

$$\mu_1(X) = \mu_2(\varphi(X))$$

for every $X \in B(E_1)$.

The proof of Lemma 3.1 can be found in [3].

The solution of the Problem 1.2 is contained in the following statement.

Theorem 3.1. *Let (G,ρ) be a Polish metric space which is dense-in-itself. Let μ be a diffused Borel probability measure defined in (G,ρ). Then there exist a metric ρ_φ and a group operation $\odot_\varphi$ in G such that the following three conditions*

(i) The class of Borel measurable subsets of G generated by the metric ρ_φ coincides with the class of Borel measurable subsets of the same space generated by the metric ρ,

(ii) $(G,\rho_\varphi,\odot_\varphi)$ is a compact Polish group

and

(iii) μ is a left (right or two-sided) invariant probability Haar measure in $(G,\rho_\varphi,\odot_\varphi)$

hold true.

Proof. Let $(G_2,\rho_2,\odot_2)$ be a compact Polish group which is dense-in-itself equipped with two-sided invariant Haar measure λ_2. By Lemma 3.1, there exists a Borel isomorphism $\varphi : (G,B(G)) \to (G_2,B(G_2))$ such that

$$\mu(X) = \lambda_2(\varphi(X))$$

for every $X \in B(G)$.

We set

$$x \odot_\varphi y = \varphi^{-1}(\varphi(x) \odot_2 \varphi(y))$$

and

$$\rho_\varphi(x,y) = \rho_2(\varphi(x),\varphi(y))$$

for $x,y \in G$.

By Theorem 2.1 we know that $(G,\odot_\varphi,\rho_\varphi)$ is a compact Polish group without isolated points which is Borel isomorphic to the compact Polish group $(G_2,\odot_2,\rho_2)$ and a measure λ_φ, defined by

$$(\forall Y)(Y \in \mathcal{B}(G_2) \to \lambda_\varphi(Y) = \lambda(\varphi^{-1}(Y))),$$

is a two-sided invariant Haar measure in G.

Since $\varphi : (G,B(G)) \to (G_2,B(G_2))$ is Borel isomorphism, we deduce that

$$\{z : \rho_\varphi(x,z) < r\} = \{z : \rho_2(\varphi(x),\varphi(z)) < r\} = \varphi^{-1}(\{w : \rho_2(\varphi(x),w) < r\}) \in B(G_2).$$

for each $x \in G$ and $r > 0$.

Since $\mathcal{B}(G)$ is σ-algebra, we deduce that $\mathcal{B}_{\rho_\varphi}(G) \subseteq \mathcal{B}(G)$.

We have to show that $\mathcal{B}(G) \subseteq \mathcal{B}_{\rho_\varphi}(G)$. Assume the contrary and let $X \in \mathcal{B}(G) \setminus \mathcal{B}_{\rho_\varphi}(G)$. Since $\varphi : (G,\mathcal{B}(G)) \to (G_2,\mathcal{B}(G_2))$ is Borel isomorphism, we deduce $\varphi(X) \in \mathcal{B}(G_2)$. Then, by Theorem 2.1 we deduce that $X \in B_{\rho_\varphi}(G)$ and we get the contradiction.

$\square$

Remark 3.1. *In the proof of Theorem 3.1, if under $(G_2,\rho_2,\odot_2)$ we take an abelian compact Polish group without isolated points and with a two-sided invariant Haar measure λ then the group $(G,\rho_\varphi,\odot_\varphi)$ will be a compact abelian Polish group without isolated points. Similarly, if under $(G_2,\rho_2,\odot_2)$ we take a*

non-abelian compact Polish group without isolated points and with a two-sided invariant Haar measure λ then the group $(G, \rho_\varphi, \odot_\varphi)$ also will be a non-abelian compact Polish group without isolated points.

The solution of Problem 1.3 is contained in the following statement.

Theorem 3.2. *Let (G, ρ) be a Polish metric space which is dense-in-itself. Let μ be a diffused σ-finite non-finite Borel measure defined in (G, ρ). Then there exist a metric ρ_φ, a group operation $\odot_\varphi$ in G and the Borel measure $\mu^\star$ in G such that the following conditions*

(i) The class of Borel measurable subsets of G generated by the metric ρ_φ coincides with the class of Borel measurable subsets of the same space generated by the metric ρ,

(ii) $(G, \rho_\varphi, \odot_\varphi)$ is a non-compact locally compact Polish group,

(iii) The measures $\mu^\star$ and μ are equivalent,

and

(iv) $\mu^\star$ is a left (right or two-sided) invariant σ-finite non-finite Haar measure in $(G, \rho_\varphi, \odot_\varphi)$

hold true.

Proof. Let $(G_2, \rho_2, \odot_2)$ be a non-compact locally compact Polish group which is dense-in-itself with two-sided invariant σ-finite non-finite Haar measure λ_2 (for example, the real axis $\mathbf{R}$ with Lebesgue measure). Let $(X_k^{(2)})_{k \in N}$ be a partition of the G_2 into Borel measurable subsets such that $0 < \lambda_2(X_k^{(2)}) < +\infty$ for $k \in N$. We set

$$\mu_2(X) = \sum_{k \in N} \frac{\lambda_2(X \cap X_k^{(2)})}{2^k \lambda_2(X_k^{(2)})}$$

for $X \in \mathcal{B}(G_2)$.

Similarly, let $(Y_k)_{k \in N}$ be a partition of the G into Borel measurable subsets such that $0 < \mu(Y_k) < +\infty$ for $k \in N$. We set

$$\mu_1(Y) = \sum_{k \in N} \frac{\mu(Y \cap Y_k)}{2^k \mu(Y_k)}$$

for $Y \in \mathcal{B}(G)$.

By Lemma 3.1, there exists a Borel isomorphism $\varphi : (G, B(G)) \to (G_2, B(G_2))$ such that

$$\mu_1(Y) = \mu_2(\varphi(Y))$$

for every $Y \in B(G)$.

We set

$$x \odot_\varphi y = \varphi^{-1}(\varphi(x) \odot_2 \varphi(y))$$

and

$$\rho_\varphi(x, y) = \rho_2(\varphi(x), \varphi(y))$$

for $x, y \in G$.

By Theorem 2.1 we know that $(G, \odot_\varphi, \rho_\varphi)$ is a locally compact non-compact Polish group without isolated points which is Borel isomorphic to the non-compact locally compact Polish group $(G_2, \odot_2, \rho_2)$.

Now we put

$$\mu^\star(X) = \sum_{k \in N} 2^k \lambda_2(X_k^{(2)})\mu_1(X \cap \varphi^{-1}(X_k^{(2)}))$$

for $X \in \mathcal{B}(G)$.

By using Theorem 2.1 and the coincidence of Borel σ-algebras $\mathcal{B}(G)$ and $\mathcal{B}_{\rho_\varphi}(G)$, we have to show only that the measure $\mu^\star$ is a two-sided invariant measure in G. Indeed, for $h_1, h_2 \in G$ and $X \in \mathcal{B}(G)$, we have

$$\mu^\star(h_1 \odot_\varphi X \odot_\varphi h_2) = \sum_{k \in N} 2^k \lambda_2(X_k^{(2)})\mu_1((h_1 \odot_\varphi X \odot_\varphi h_2) \cap \varphi^{-1}(X_k^{(2)})) =$$

$$\sum_{k \in N} 2^k \lambda_2(X_k^{(2)})\mu_2(\varphi[(h_1 \odot_\varphi X \odot_\varphi h_2) \cap \varphi^{-1}(X_k^{(2)}]) =$$

$$\sum_{k \in N} 2^k \lambda_2(X_k^{(2)}) \sum_{i \in N} \frac{\lambda_2(\varphi[(h_1 \odot_\varphi X \odot_\varphi h_2) \cap \varphi^{-1}(X_k^{(2)})] \cap X_i^{(2)})}{2^i \lambda_2(X_i^{(2)})} =$$

$$\sum_{k \in N} 2^k \lambda_2(X_k^{(2)}) \sum_{i \in N} \frac{\lambda_2(\varphi[\varphi^{-1}\{\varphi h_1 \odot \varphi(X) \odot \varphi(h_2)\} \cap \varphi^{-1}(X_k^{(2)})] \cap X_i^{(2)})}{2^i \lambda_2(X_i^{(2)})} =$$

$$\sum_{k \in N} 2^k \lambda_2(X_k^{(2)}) \sum_{i \in N} \frac{\lambda_2(((\varphi h_1 \odot \varphi(X) \odot \varphi(h_2)) \cap X_k^{(2)}) \cap X_i^{(2)})}{2^i \lambda_2(X_i^{(2)})} =$$

$$\sum_{k \in N} \lambda_2((\varphi h_1 \odot \varphi(X) \odot \varphi(h_2)) \cap X_k^{(2)}) =$$

$$\lambda_2(\varphi h_1 \odot \varphi(X) \odot \varphi(h_2)) = \lambda_2(\varphi(X)) = \sum_{k \in N} 2^k \lambda_2(X_k^{(2)})\mu_2(\varphi(X) \cap X_k^{(2)}) =$$

$$\sum_{k \in N} 2^k \lambda_2(X_k^{(2)})\mu_1(\varphi^{-1}[\varphi(X) \cap X_k^{(2)}]) =$$

$$\sum_{k \in N} 2^k \lambda_2(X_k^{(2)})\mu_1(X \cap \varphi^{-1}(X_k^{(2)})) = \mu^\star(X).$$

$\square$

Remark 3.2. *The result of Theorem 3.2 remains true if μ is a diffused Borel probability measure in (G, ρ).*

As a simple consequence of Theorem 3.2, we have the following corollary.

Corollary 3.1. *Let (G, ρ) be a Polish metric space which is dense-in-itself. Let μ be a diffused σ-finite non-finite Borel measure defined in (G, ρ). Then there exist a metric ρ_φ and a group operation $\odot_\varphi$ in G such that the following three conditions*

(i) The class of Borel measurable subsets of G generated by the metric ρ_φ coincides with the class of Borel measurable subsets of the same space generated by the metric ρ,

(ii) $(G, \rho_\varphi, \odot_\varphi)$ is a non-compact locally compact Polish group

and

(iii) The measure μ is a two-sided quasi-invariant [4] Borel probability measure in $(G, \rho_\varphi, \odot_\varphi)$

hold true.

Finally, we state the following problem

Problem 3.1 Let (G, ρ) be a Polish metric space which is dense-in-itself, that is, G is a space homeomorphic to a separable complete metric space and G has no isolated points. Let μ be a diffused non-finite σ-finite Borel measure defined in (G, ρ). Do there exist a metric ρ_1 and a group operation $\odot$ in G such that the following three conditions

(j) The class of Borel measurable subsets of G generated by the metric ρ_1 coincides with the class of Borel measurable subsets of the same space generated by the metric ρ,

(jj) $(G, \rho_1, \odot)$ is a non-compact locally compact Polish group

and

(jjj) μ is a left(right or two-sided) invariant non-finite σ-finite Haar measure in $(G, \rho_1, \odot)$

hold true ?

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